SYMPLECTIC ACTIONS ON COADJOINT ORBITS

H. ARATYN [a,1], E. NISSIMOV [b], S. PACHEVA [b] and A.H. ZIMERMAN [a,2]

[a] *Department of Physics, University of Illinois at Chicago, Box 4348, Chicago, IL 60680, USA*

[b] *Institute of Nuclear Research and Nuclear Energy, Boulevard Lenin 72, 1784 Sofia, Bulgaria*

Received 10 January 1990

We present a compact expression for the field theoretical actions based on the symplectic analysis of coadjoint orbits of Lie groups. The final formula for the action density α_c becomes a bilinear form $\langle (S, 1/\lambda), (y, m_y) \rangle$, where S is a 1-cocycle of the Lie group (a schwarzian type of derivative in conformal case), λ is a coefficient of the central element of the algebra and $\mathcal{Y} \equiv (y, m_y)$ is the generalized Maurer–Cartan form. In this way the action is fully determined in terms of the basic group theoretical objects. This result is illustrated on a number of examples, including the superconformal model with $N=2$. In this case the method is applied to derive the $N=2$ superspace generalization of the $D=2$ Polyakov (super-)gravity action in a manifest (2, 0) supersymmetric form. As a byproduct we also find a natural (2, 0) superspace generalization of the Beltrami equations for the (2, 0) supersymmetric world-sheet metric describing the transition from the "conformal" to the "chiral" gauge.

1. Actions on the coadjoint orbits for groups with central extension

Recently, symplectic methods have been employed for the purpose of constructing actions of dynamical systems on the coadjoint orbits of a Lie group G [1–3]. For the fixed covector U one defines the coadjoint orbit $\mathcal{O}_U$ of G on which there exists a canonical symplectic 2-form [4,5] which is given by

$$\Omega_U = \tfrac{1}{2} \langle U, [\mathcal{Y}, \mathcal{Y}] \rangle \,, \tag{1}$$

where U is a generic point on $\mathcal{O}_U$, $\mathcal{Y}$ is a 1-form taking values in the Lie algebra of G and $\langle \, , \, \rangle$ is an invariant bilinear form. In this recipe $\mathcal{Y}$ is obtained as a solution to the following basic equation:

$$\mathrm{d}U = \mathrm{ad}^*_{\mathcal{Y}} \, U \,. \tag{2}$$

The infinitesimal coadjoint representation $\mathrm{ad}^*_{\mathcal{Y}}$ is defined in terms of the infinitesimal adjoint transformation

$$\langle \mathrm{ad}^*_{\mathcal{Y}} U, X \rangle = -\langle U, \mathrm{ad}_{\mathcal{Y}} X \rangle = -\langle U, [\mathcal{Y}, X] \rangle . \tag{3}$$

It follows from definition (2) that $\mathcal{Y}$ obeys the Maurer–Cartan equation. Observe namely that, since $\mathrm{d}^2 U = 0$ we have for the fixed element X of the Lie algebra of G

$$\mathrm{d} \langle U, [\mathcal{Y}, X] \rangle = 0 \tag{4}$$

or

$$\langle \mathrm{ad}^*_{\mathcal{Y}} U, [\mathcal{Y}, X] \rangle = -\langle U, [\mathrm{d}\mathcal{Y}, X] \rangle \, ; \tag{5}$$

using the Jacobi identity and eq. (2) we can identify $\mathrm{d}\mathcal{Y}$ to be

$$\mathrm{d}\mathcal{Y} = \tfrac{1}{2} [\mathcal{Y}, \mathcal{Y}] \,, \tag{6}$$

which is our basic Maurer–Cartan equation.

The symplectic form Ω_U is closed, hence (locally) exact:

$$\Omega_U = \mathrm{d}\alpha \,. \tag{7}$$

The simplest action, describing the dynamical system defined by Ω_U and having the orbit $\mathcal{O}_U$ as its phase space, takes the form

$$\mathcal{A} = \int \alpha \,, \tag{8}$$

where the integral is over a curve on the orbit $\mathcal{O}_U$.

Thus, finding the geometric action (8) equals the problem of solving eq. (2) for the 1-form $\mathcal{Y}$. This is

[1] Work supported in part by US Department of Energy, contract DE-FG02-84ER40173.

[2] On leave of absence from Instituto de Fisica Teórica – UNESP, partially supported by CNPq (Brazil).

achieved by parametrizing the elements U of the orbit $\mathscr{C}_{U_0}$ through the group variables $g\in G$ as

$$U\equiv U(g)=\mathrm{Ad}_g^*\, U_0\ , \tag{9}$$

where U_0 is a fixed generic point of this orbit. With this parametrization, eq. (2) looks as

$$\mathrm{d}(\mathrm{Ad}^*(g)U_0)=\mathrm{ad}^*_{\mathscr{Y}}(\mathrm{Ad}_g^*\, U_0). \tag{10}$$

From now on, we deal only with groups with central extensions; let $\tilde{\mathrm{G}}$ be the central extension of G. Accordingly the elements of the corresponding Lie algebra are represented by pairs (a, n), where a is in the Lie algebra of G while n is a central element. The dual vector is written as (B, c).

We describe the bilinear form $\langle\ ,\ \rangle$ on $\tilde{\mathrm{G}}$ in terms of the bilinear form $\langle\ ,\ \rangle_0$ on G as follows:

$$\langle(\ ,c),(\ ,n)\rangle=\langle\ ,\ \rangle_0+cn\ . \tag{11}$$

We now assume that the adjoint transformation by $g\in\mathrm{G}$ on the (a, n) pair takes the following general form:

$$\mathrm{Ad}_g(a,n)=(g\circ a,\, n+\lambda\langle S(g^{-1}),a\rangle_0)\ , \tag{12}$$

where $g\circ a$ defines the standard adjoint transformation on G.

By invariance of the bilinear form eq. (12) leads to the following coadjoint transformation:

$$\mathrm{Ad}_g^*(B,c)=(g\circ{}^*B+c\lambda S(g),c)\ , \tag{13}$$

where $\circ^*$ denotes the coadjoint action for the group G.

One easily verifies that S must satisfy the following cocycle condition [4]:

$$(\delta S)(g_1,g_2)=g_1\circ{}^*S(g_2)-S(g_1g_2)+S(g_1)=0\ , \tag{14}$$

as well as relations $S(I)=0$ and $S(g)=-g\circ{}^*S(g^{-1})$ in order to ensure the group property of Ad_g in (12). The adjoint representation of the Lie group induces the adjoint representation of its Lie algebra,

$$\mathrm{ad}_{(a_1,\,n_1)}(a_2,n_2)=[(a_1,n_1),(a_2,n_2)]\ , \tag{15}$$

where the commutator of the Lie algebra is given by

$$[(a_1,n_1),(a_2,n_2)]=([a_1,a_2],\,-\lambda\langle s(a_1),a_2\rangle_0)\ ; \tag{16}$$

here $s(a)$ is the infinitesimal limit of $S(g)$ which defines the Lie algebra cocycle in the above formula.

Using eq. (3) one can find the corresponding coadjoint action

$$\mathrm{ad}^*_{(a,n)}(B,c)=(\mathrm{ad}^*_a(B)+c\lambda s(a),0)\ . \tag{17}$$

Inserting $U_0=(B_0,\, c)$ and the above definitions we find by comparing terms linear in c that

$$\mathrm{d}(s,\,1/\lambda)=\mathrm{ad}^*_{(y,\ m_y)}(S,\,1/\lambda) \tag{18}$$

with $\mathscr{Y}=(y,\, m_y)$. One sees that for the group with the central extension the pair $(S,\,1/\lambda)$ becomes an element of the coadjoint orbit, while S was a covector of the original group G. One easily verifies that indeed $(S,\,1/\lambda)$ transforms according to the coadjoint action (13)

$$(S(g_2),1/\lambda)\xrightarrow{g_2\to g_1g_2}(S(g_1g_2),1/\lambda)$$
$$=\mathrm{Ad}^*_{g_1}(S(g_2),1/\lambda)\ , \tag{19}$$

as follows from the cocycle condition (14).

We will now prove the main result of this paper. We propose the following compact expression for the c-dependent part of the action density as defined in (8):

$$\alpha_c=-\lambda c\langle(S,\,1/\lambda),\mathscr{Y}\rangle\ . \tag{20}$$

We prove this identity by showing that acting with the exterior derivative we recover Ω_c; the c-dependent part of (7).

$$\mathrm{d}\alpha_c=-\lambda c\langle \mathrm{d}(S,\,1/\lambda),\mathscr{Y}\rangle-\lambda c\langle(S,\,1/\lambda),\mathrm{d}\mathscr{Y}\rangle\ ; \tag{21}$$

in components we have $\mathrm{d}\mathscr{Y}=(\mathrm{d}y,\,\mathrm{d}m_y)$. Recall that from (3), (15) and (18) we have an identity

$$\langle \mathrm{d}(S,1/\lambda),\mathscr{Y}\rangle=-\langle(S,1/\lambda),[\mathscr{Y},\mathscr{Y}]\rangle$$
$$=-2\langle(S,1/\lambda),\mathrm{d}\mathscr{Y}\rangle\ , \tag{22}$$

which leads to

$$\mathrm{d}\alpha_c=-\tfrac{1}{2}\lambda c\langle \mathrm{d}(S,1/\lambda),\mathscr{Y}\rangle=-\tfrac{1}{2}\lambda c\langle \mathrm{d}S,y\rangle_0\ , \tag{23}$$

which provides an alternative and useful formula for derivation of α_c by factoring out the exterior derivative d.

We can verify that eq. (23) reproduces the c-dependent part Ω_c of the symplectic 2-form Ω_U by inserting $U=\mathrm{Ad}_g^*(B_0,c)$ into expression (1) and collecting the terms proportional to c. A short calculation gives the desired result

$$\Omega_c = \tfrac{1}{2}\langle (c\lambda S, c), \mathrm{ad}_{\mathscr{Y}}\, \mathscr{Y}\rangle = -\tfrac{1}{2}\lambda c\langle \mathrm{d}(S, 1/\lambda), \mathscr{Y}\rangle , \tag{24}$$

where we used relations (3), (15) and (18).

Note that in components the expression for the action density given by (20) reads

$$\alpha_c = -\lambda_c[\langle S, y\rangle_0 + (1/\lambda) m_y] , \tag{25}$$

where the central element m_y of $\mathscr{Y}$ is fixed by the Maurer–Cartan equation (6) and explicitly given by

$$\mathrm{d}m_y = \tfrac{1}{2}\lambda\langle s(y), y\rangle_0 . \tag{26}$$

Let us now concentrate on the remaining (c-independent) part of Ω_U,

$$\Omega_{B_0} = \tfrac{1}{2}\langle \mathrm{Ad}^*_g(B_0, 0), [\mathscr{Y}, \mathscr{Y}]\rangle . \tag{27}$$

From the alternative expressions for Ω_{B_0}

$$\begin{aligned}\Omega_{B_0} &= \langle \mathrm{Ad}^*_g(B_0, 0), \mathrm{d}\mathscr{Y}\rangle \\ &= -\tfrac{1}{2}\langle \mathrm{d}\,\mathrm{Ad}^*_g(B_0, 0), \mathscr{Y}\rangle ,\end{aligned} \tag{28}$$

we easily prove that

$$\Omega_{B_0} = -\mathrm{d}(\langle \mathrm{Ad}^*_g(B_0, 0), \mathscr{Y}\rangle) , \tag{29}$$

from which it follows that

$$\alpha_{B_0} = -\langle \mathrm{Ad}^*_g(B_0, 0), \mathscr{Y}\rangle . \tag{30}$$

Finally, we collect the two contributions α_{B_0} and α_c into the total action density α:

$$\alpha = \alpha_{B_0} + \alpha_c = -\langle \mathrm{Ad}^*_g(B_0, c), \mathscr{Y}\rangle . \tag{31}$$

From the group symmetry

$$\mathrm{Ad}^*_{gh}(B_0, c) = \mathrm{Ad}^*_g(B_0, c) \tag{32}$$

under the action of the elements $h \in \mathrm{H}$ of the stationary subgroup H, one derives the corresponding invariance of the action $\mathscr{A}$ in (8).

2. Examples

2.1. Kac–Moody

Following refs. [2,6] we denote elements of the Kac–Moody algebra by pairs $(u(x), m)$, while the dual space elements are $(v(x), -k)$. The natural bilinear form $\langle\ ,\ \rangle_0$ is

$$\langle v, u\rangle_0 = \mathrm{Tr}\int uv\, \mathrm{d}x , \tag{33}$$

where we integrate over S^1.

The transformation $g\circ u$ is naturally given by gug^{-1} and the trace property gives $g\circ{}^*v = gvg^{-1}$.

The first step in this example is the derivation of the Maurer–Cartan form for which we use eq. (10). This gives

$$\mathscr{Y} = (y, m_y) = (\mathrm{d}g\, g^{-1}, m_y) . \tag{34}$$

In order to find the central element m_y we use the component version of the Maurer–Cartan equation (26) arriving at

$$\begin{aligned}\mathrm{d}m_y &= \tfrac{1}{2}\lambda\langle s(y), y\rangle_0 \\ &= -\tfrac{1}{2}\lambda\, \mathrm{d}\,\mathrm{Tr}\int \{g^{-1}\, \mathrm{d}g\, g^{-1}\, \partial g \\ &\quad + \mathrm{d}^{-1}[(\mathrm{d}g\, g^{-1})^2\, \partial g\, g^{-1}]\} .\end{aligned} \tag{35}$$

The next step is to solve eq. (18) for $S(g)$ upon inserting there the result (34) $\mathscr{Y} = (y, m_y)$. One easily finds

$$S(g) = \partial g\, g^{-1} . \tag{36}$$

Eq. (36) together with the normalization $\lambda = -1/2\pi$ fully specifies the adjoint and coadjoint transformations in (12) and (13).

Inserting into expression (25) the above values of S, y and m_y we obtain the Wess–Zumino–Novikov–Witten action [2]

$$\begin{aligned}\alpha_k = -\frac{k}{4\pi}\mathrm{Tr}\int & [g^{-1}\, \mathrm{d}g\, g^{-1}\, \partial g \\ & - \mathrm{d}^{-1}((\mathrm{d}g\, g^{-1})^2\, \partial g\, g^{-1})] .\end{aligned} \tag{37}$$

From eq. (30) one easily obtains the remaining part $\alpha_{v_0} = -\mathrm{Tr}\int(v_0 g^{-1}\mathrm{d}g)$ of the total action density.

2.2. Virasoro algebra

In the case of the Virasoro algebra, the bilinear form for the pair $(g(x), n)$ and the dual $(b(x), c)$ is defined in terms of [2,7,8]

$$\langle b, g\rangle_0 = \int b(x) g(x)\, \mathrm{d}x . \tag{38}$$

Along the lines of discussion in the previous section we specify the fundamental objects for this case. The adjoint transformation $g\circ u$ is now described by the reparametrization $x \to F(x)$ for $x \in S^1$ and $F \in \mathrm{diff}\, S^1$, explicitly we have $g(x) \to [1/F'(x)] \times g(F(x))$ following from the fact that the vector field is $g(x)\,\partial/\partial x$. Therefore the coadjoint action is given by $b(x) \to F'(x)^2 b(F(x))$.

The coefficient of the central element of the Virasoro algebra dictates the choice of $\lambda = -1/24\pi$.

The 1-cocycle of the conformal group is given by the schwarzian derivative

$$S(F) = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 . \tag{39}$$

In this way we have completely determined the adjoint and coadjoint transformations in (12) and (13) for the Virasoro case.

The Maurer–Cartan form is obtained from eq. (10):

$$\mathscr{Y}(F) = (y(F), m_y) = \left(\frac{\mathrm{d}F}{F'}, m_y\right) \tag{40}$$

with the central term m_y derived from (26):

$$\mathrm{d}m_y = \frac{1}{48\pi}\int y''' y = \frac{\mathrm{d}}{48\pi}\int\left[\frac{F'''}{F'} - \left(\frac{F''}{F'}\right)^2\right] y(F) . \tag{41}$$

Substituting our expression for S, $\mathscr{Y}$ and λ into eq. (25) we obtain [2]

$$\alpha_c = \frac{c}{48\pi}\int\left[\frac{F'''}{F'} - 2\left(\frac{F''}{F'}\right)^2\right]\frac{\mathrm{d}F}{F'} . \tag{42}$$

2.3. Superconformal models, $N=2$ case

Let us make a brief recapitulation of the structure of the N-extended superspace [9]. We firstly introduce coordinates $z = (x, \theta^i)$, $i = 1, 2, \ldots, N$, with the left supercovariant derivative $\mathrm{D}^i = \partial_{\theta^i} + \theta^i\,\partial_x$ satisfying

$$\{\mathrm{D}^i, \mathrm{D}^j\} = 2\delta^{ij}\partial_x . \tag{43}$$

The action of the superconformal group is given by the analytic transformations

$$z = (x, \theta^i) \to \tilde{z} = (\tilde{x}, \tilde{\theta}^i), \tag{44}$$

subject to the constraint

$$\mathrm{D}^i = (\mathrm{D}^i\tilde{\theta}^j)\tilde{\mathrm{D}}^j , \tag{45}$$

leading to

$$\mathrm{D}^i\tilde{x} = \tilde{\theta}^j \mathrm{D}^i\tilde{\theta}^j , \tag{46}$$

$$(\mathrm{D}^i\tilde{\theta}^j)(\mathrm{D}^k\tilde{\theta}^j) = \delta^{ik}(\partial_x\tilde{x} + \tilde{\theta}^j\,\partial_x\tilde{\theta}^j) , \tag{47}$$

$$\det\left(\frac{\mathrm{D}\tilde{\theta}}{(\partial_x\tilde{x} + \tilde{\theta}^j\,\partial_x\tilde{\theta}^j)^{1/2}}\right) = \pm 1 . \tag{48}$$

This constraint can be solved in terms of the infinitesimal unrestricted superfield $\epsilon(z)$ [9]:

$$\delta x = \epsilon - \tfrac{1}{2}\theta^i\,\mathrm{D}^i\epsilon , \quad \delta\theta^i = \tfrac{1}{2}\mathrm{D}^i\epsilon . \tag{49}$$

Extending the case of the Virasoro algebra to N-extended superspace we introduce the pair $(g(x, \theta^i), n)$. The dual $(B(x, \theta^i), c)$ is defined according to the following bilinear form:

$$\langle (B, c), (g, n)\rangle = \langle B, g\rangle_0 + cn, \tag{50}$$

with $\langle B, g\rangle_0 = \int \mathrm{d}z\, B(x, \theta) g(x, \theta)$.

The adjoint action $g \circ u$ can be generally described for all N as

$$g(x, \theta^i) \to [\det(\mathrm{D}\tilde{\theta})]^{-2/N} g(\tilde{x}, \tilde{\theta}^i) , \tag{51}$$

while the coadjoint action $g \circ {}^*B$ takes the form

$$B(x, \theta^i) \to [\det(\mathrm{D}\tilde{\theta})]^{(4-N)/N} B(\tilde{x}, \tilde{\theta}^i) . \tag{52}$$

We are now ready to determine a Maurer–Cartan form y for an arbitrary N. We use an obvious identification $U_0 = (B_0(x, \theta^i), c)$ in eq. (10) and isolate the terms $\partial_{\tilde{x}}B_0$. The calculation is based on the identity

$$\mathrm{d}B_0 = \partial_{\tilde{x}}B_0\,\delta\tilde{l} + \mathrm{d}\tilde{\theta}^i\,\tilde{\mathrm{D}}^i B_0 , \tag{53}$$

where for convenience we introduced the symbol

$$\delta\tilde{l} \equiv \mathrm{d}\tilde{x} + \tilde{\theta}^i\,\mathrm{d}\tilde{\theta}^i . \tag{54}$$

Furthermore, using eq. (43) and the constraint (45) we find the following technical identity:

$$\partial_x = (\partial_x\tilde{\theta}^k)\,\tilde{\mathrm{D}}^k + [\det(\mathrm{D}\tilde{\theta})]^{2/N}\,\partial_{\tilde{x}} . \tag{55}$$

Now comparing on both sides of (10) the terms in $\partial_{\tilde{x}}B_0$ we obtain the group component of the Maurer–Cartan form,

$$y = \frac{\delta\tilde{l}}{[\det(\mathrm{D}\tilde{\theta})]^{2/N}} . \tag{56}$$

What remains to complete our construction for the general superconformal models is to discuss the corresponding 1-cocycles. They turn out to be the superschwarzian derivatives listed in ref. [9] for $0 \leqslant N \leqslant 4$. The cocycle condition (14) translates in the case of superconformal transformations to [9]

$$S^N(z, \bar{\bar{z}}) = [\det(\mathrm{D}\tilde{\theta})]^{(4-N)/N} S^N(\tilde{z}, \bar{\bar{z}}) + S^N(z, \tilde{z}) . \tag{57}$$

Let us now discuss in greater detail the special example of $N=2$ (for the discussion of $N=1$ we refer to refs. [10, 11]). It is convenient here to choose the complex basis [12,13] $\theta = (\theta^1 + \mathrm{i}\theta^2)/\sqrt{2}$, $\bar{\theta} = (\theta^1 - \mathrm{i}\theta^2)/\sqrt{2}$, with the supercovariant derivatives $\mathrm{D} = \partial_{\bar{\theta}} + \theta\,\partial_x$ and $\bar{\mathrm{D}} = \partial_\theta + \bar{\theta}\,\partial_x$. Note, that $\mathrm{D}^2 = \bar{\mathrm{D}}^2 = 0$, $\{\mathrm{D}, \bar{\mathrm{D}}\} = 2\partial_x$ and the superconformal condition (45) allows the choice

$$\mathrm{D}\tilde{\theta} = 0 , \quad \bar{\mathrm{D}}\tilde{\bar{\theta}} = 0 . \tag{58}$$

We also rewrite in this notation the infinitesimal transformation (49):

$$\delta x = \epsilon - \tfrac{1}{2}(\theta\bar{\mathrm{D}} + \bar{\theta}\mathrm{D})\epsilon, \quad \delta\theta = \tfrac{1}{2}\mathrm{D}\epsilon, \quad \delta\bar{\theta} = \tfrac{1}{2}\bar{\mathrm{D}}\epsilon. \tag{59}$$

In this basis the adjoint and coadjoint actions of the extended $N=2$ superVirasoro group look as

$$\mathrm{Ad}_{\tilde{z}}(g, n) = \left(\frac{g(\tilde{x}, \tilde{\theta}, \tilde{\bar{\theta}})}{(\bar{\mathrm{D}}\tilde{\theta})(\mathrm{D}\tilde{\bar{\theta}})} , \right.$$
$$\left. n - \frac{1}{8\pi} \int \mathrm{d}x\, \mathrm{d}\theta\, \mathrm{d}\bar{\theta}\, S(z; \tilde{z}^{-1}) g(x, \theta, \bar{\theta}) \right) , \tag{60}$$

$$\mathrm{Ad}^*_{\tilde{z}}(B, c) = \left((\bar{\mathrm{D}}\tilde{\theta})(\mathrm{D}\tilde{\bar{\theta}}) B(\tilde{x}, \tilde{\theta}, \tilde{\bar{\theta}}) - \frac{c}{8\pi} S(z; \tilde{z}), c \right). \tag{61}$$

In the above equations $S(z; \tilde{z})$ is the superschwarzian derivative

$$S(z;\tilde{z}) = \frac{\partial_x \mathrm{D}\tilde{\bar{\theta}}}{\mathrm{D}\tilde{\bar{\theta}}} - \frac{\partial_x \bar{\mathrm{D}}\tilde{\theta}}{\bar{\mathrm{D}}\tilde{\theta}} - 2\frac{\partial_x \tilde{\theta}\, \partial_x \tilde{\bar{\theta}}}{(\mathrm{D}\tilde{\bar{\theta}})(\bar{\mathrm{D}}\tilde{\theta})} \tag{62}$$

and the coefficient λ is $\lambda = -1/8\pi$. Note, that in going to the complex basis use was made of the following $N=2$ relation:

$$(\mathrm{D}\tilde{\bar{\theta}})(\bar{\mathrm{D}}\tilde{\theta}) = \det(\mathrm{D}\tilde{\theta}) . \tag{63}$$

The Maurer–Cartan 1-form y becomes now

$$y = \frac{\delta \tilde{I}}{(\mathrm{D}\tilde{\bar{\theta}})(\bar{\mathrm{D}}\tilde{\theta})} . \tag{64}$$

To derive the action it is convenient to work with eq. (23) yielding

$$\mathrm{d}\alpha_c = \frac{c}{16\pi} \int \mathrm{d}x\, \mathrm{d}\theta\, \mathrm{d}\bar{\theta}\, \mathrm{d}S\, y \tag{65}$$

with S, y as in (62), (64). Making use of integrations by parts and of the relations $\mathrm{d}(\delta\tilde{I}) = 2\,\mathrm{d}\tilde{\theta}\,\mathrm{d}\tilde{\bar{\theta}}$, $\mathrm{D}(\delta\tilde{I}) = 2\,\mathrm{D}\tilde{\bar{\theta}}\,\mathrm{d}\tilde{\theta}$ and $\bar{\mathrm{D}}(\delta\tilde{I}) = 2\,\bar{\mathrm{D}}\tilde{\theta}\,\mathrm{d}\tilde{\bar{\theta}}$, it is not difficult to extract the total exterior derivative from the integrand in (65):

$$\alpha_c = -\frac{c}{8\pi} \int \mathrm{d}x\, \mathrm{d}\theta\, \mathrm{d}\bar{\theta} \frac{\partial_x \tilde{\theta}\, \mathrm{d}\tilde{\bar{\theta}} - \partial_x \tilde{\bar{\theta}}\, \mathrm{d}\tilde{\theta}}{(\mathrm{D}\tilde{\bar{\theta}})(\bar{\mathrm{D}}\tilde{\theta})} . \tag{66}$$

This term enters the total action together with α_{B_0} (30),

$$\alpha_{B_0} = -\int \mathrm{d}x\, \mathrm{d}\theta\, \mathrm{d}\bar{\theta}\, B_0\, \delta\tilde{I}. \tag{67}$$

This result generalizes the $D=2$ Polyakov (super)gravity action [14] in a manifest (2, 0) supersymmetric form:

$$W^{N=2}_{\mathrm{SP}} = -\int \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}\theta\, \mathrm{d}\bar{\theta} \Bigg(B_0(t, \tilde{z})$$
$$\times (\partial_t \tilde{x} + \tilde{\theta}\, \partial_t \tilde{\bar{\theta}} + \tilde{\bar{\theta}}\, \partial_t \tilde{\theta})$$
$$+ \frac{c}{8\pi} \frac{\partial_x \tilde{\theta}\, \partial_t \tilde{\bar{\theta}} - \partial_x \tilde{\bar{\theta}}\, \partial_t \tilde{\theta}}{(\mathrm{D}\tilde{\bar{\theta}})(\bar{\mathrm{D}}\tilde{\theta})} \Bigg) , \tag{68}$$

where t indicates the parameter of the curve in the general eq. (8).

3. (2, 0) supersymmetric Beltrami equations

The following superconformal reparametrization on the (2, 0) superspace:

$$(t, x, \theta, \bar{\theta}) \to (x^+, x^-, \phi, \bar{\phi}) ,$$
$$x^+ = t , \quad x^- = \tilde{x} = \tilde{x}(t, x, \theta, \bar{\theta}) ,$$
$$\phi = \tilde{\theta} = \tilde{\theta}(t, x, \theta, \bar{\theta}) , \quad \bar{\phi} = \tilde{\bar{\theta}} = \tilde{\bar{\theta}}(t, x, \theta, \bar{\theta}) , \tag{69}$$

is being induced by the superconformal group transformation (44). Let us also introduce the inverse superconformal reparametrization with respect to (69):

$$t = x^+ , \quad x = f(x^+, x^-, \phi, \bar{\phi}) ,$$
$$\theta = \psi(x^+, x^-, \phi, \bar{\phi}) , \quad \bar{\theta} = \bar{\psi}(x^+, x^-, \phi, \bar{\phi}) . \tag{70}$$

The inverse transformation (70) is subject to constraints of the same form as (45)–(47) for fixed x^+ [see eq. (75) below].

Now, in complete analogy with the purely bosonic case [2] and the (1, 0) supersymmetric case [15], one can show that eqs. (70) describe precisely the transition from the "conformal" to the "chiral" [14] gauge for the (2, 0) supersymmetric world-sheet metric on the (2, 0) superspace.

Indeed let us consider the following conformal form of the (2, 0) supersymmetric world-sheet metric in terms of the local coordinates $(t, x, \theta, \bar{\theta})$:

$$\mathrm{d}s^2 = (\bar{\mathrm{D}}\tilde{\theta})(\mathrm{D}\bar{\tilde{\theta}})(\mathrm{d}x + \bar{\theta}\,\mathrm{d}\theta + \theta\,\mathrm{d}\bar{\theta})\,\mathrm{d}t\,, \tag{71}$$

where the functions $\tilde{\theta}$, $\bar{\tilde{\theta}}$ are the same as in (69). It is straightforward to show that, upon performing the inverse superconformal transformation (70) and taking into account the constraints (46), (47), the metric (71) acquires the following "chiral-gauge" form in terms of the new local coordinates $(x^+, x^-, \phi, \bar{\phi})$:

$$\mathrm{d}s^2 = (\mathrm{d}x^- + \bar{\phi}\,\mathrm{d}\phi + \phi\,\mathrm{d}\bar{\phi})\,\mathrm{d}x^+ + H_{++}(\mathrm{d}x^+)^2\,, \tag{72}$$

$$H_{++} = (\partial_+ f + \bar{\psi}\,\partial_+\psi + \psi\,\partial_+\bar{\psi}) \times (\partial_- f + \bar{\psi}\,\partial_-\psi + \psi\,\partial_-\bar{\psi})^{-1}\,. \tag{73}$$

It was found already in ref. [13] (at least in the linearized case) that all geometric constraints in the (2, 0) superzweibeins can be solved in terms of six unconstrained (2, 0) superfields [i.e. (2, 0) prepotentials]. Using the residual gauge symmetries one can fix all but one of these prepotentials – the conformal prepotential H_{++}, which is naturally identified with the superfield (73) entering (72). Rewriting (73) in the form

$$\partial_+ f + \bar{\psi}\,\partial_+\psi + \psi\,\partial_+\bar{\psi} - H_{++}(\partial_- f + \bar{\psi}\,\partial_-\psi + \psi\,\partial_-\bar{\psi}) = 0 \tag{74}$$

and recalling the constraints on f, ψ, $\bar{\psi}$ [cf. eqs. (45), (47)]:

$$\tilde{\mathrm{D}} f - \psi\,\tilde{\mathrm{D}}\bar{\psi} = 0\,, \quad \bar{\tilde{\mathrm{D}}} f - \bar{\psi}\,\bar{\tilde{\mathrm{D}}}\psi = 0\,,$$

$$\tilde{\mathrm{D}}\psi = 0\,, \quad \bar{\tilde{\mathrm{D}}}\bar{\psi} = 0 \tag{75}$$

(where $\tilde{\mathrm{D}} \equiv \partial/\partial\bar{\phi} + \phi\,\partial_-$, $\bar{\tilde{\mathrm{D}}} \equiv \partial/\partial\phi + \bar{\phi}\,\partial_-$), one can naturally identify (74) as a (2, 0) superspace generalization of the Beltrami equations for the (2, 0) supersymmetric world-sheet metric. H_{++} is the (2, 0) supersymmetric analog of the Beltrami differential describing the transition between the "chiral" and "conformal" gauges (for discussions of the (1, 0) and (1, 1) supersymmetric cases we refer to ref. [15] and to ref. [16], respectively) [#1].

Acknowledgement

One of us (E.M.) gratefully acknowledges the hospitality of Professor J. Ellis at CERN and Consiglio Nazionalle delle Ricerche and Professor G. Jona-Lasinio at Rome University "La Sapienza", where part of this work was done. A.H.Z. acknowledges University of Illinois at Chicago for the hospitality and Fapesp (São Paulo, Brazil) for financial help.

[#1] One of us (E.N.) thanks F. Gieres for discussions on this point.

References

[1] A. Alekseev, L.D. Faddeev and S.L. Shatashvili, J. Geom. Phys., in press.
[2] A. Alekseev and S. L. Shatashvili, Nucl. Phys. B 323 (1989) 719.
[3] P.B. Wiegmann, Nucl. Phys. B 323 (1989) 311.
[4] A.A. Kirillov, Elements of the theory of representations (Springer, Berlin, 1976);
B. Kostant, Lecture Notes in Mathematics, Vol. 170 (Springer, Berlin, 1970) p. 87.
[5] A.T. Fomenko, Integrability and nonintegrability in geometry and mechanics (Kluwer, Deventer, 1988).
[6] A. Pressley and G. Segal, Loop groups (Oxford U.P., Oxford, 1986).
[7] E. Witten, Commun. Math. Phys. 114 (1988) 1.
[8] I. Bakas, Nucl. Phys. B 302 (1988) 189.
[9] K. Schoutens, Nucl. Phys. B 295 (1988) 634.
[10] S. Aoyama, Phys. Lett. B 228 (1989) 355.
[11] H. Aratyn, E. Nissimov, S. Pacheva and S. Solomon, Phys. Lett. B 234 (1990) 307.
[12] J.D. Cohn, Nucl. Phys. B 284 (1987) 349;
E.B. Kiritsis, Phys. Rev. D 36 (1987) 3048.
[13] R. Brooks, F. Muhammad and S. Gates, Nucl. Phys. B 268 (1986) 599; Class. Quantum Grav. 5 (1988) 785.
[14] A.M. Polyakov, Mod. Phys. Lett. A 2 (1987) 893;
V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. A 3 (1988) 819.
[15] S. Gates and F. Gieres, Nucl. Phys. B 320 (1989) 310;
F. Gieres, Gen. Rel. Grav., to appear.
[16] L. Baulieu, M. Bellon and R. Grimm, Nucl. Phys. B 321 (1989) 697.